Unification of Kinematic Wave Propagation Model of Traffic with Fundamental Diagram of Traffic Flow (June 2014)

Anush Badii

California Department of Transportation

The primary focus of this paper is to demonstrate that the diagram known as the "Fundamental Diagram of Traffic Flow", first proposed some 50 years ago by Newell, and widely referenced in the most prevalent studies of traffic engineering, can be analytically derived, and unified with the kinematic wave propagation of traffic known as LWR model, first proposed by disparate researchers more than 60 years ago, by adhering to a simple tenet that the behavior of the driver in confines of the freeway is predictable. We will also derive an expression for the backward travelling wave of the fluid dynamics model of traffic commonly known as the phantom wave consistent with observation. Also we will apply the same calculus to everyday traffic engineering problems to replicate complete highway traffic flow including the roadway capacity and demonstrate the ability to predict traffic in situations of lane convergence, on-ramp influx, lane drop and lane additions. We will show that instability of traffic flow arises out of this very predictable behavior proving that adhering to the behavioral tenet provides a model for the entire traffic flow field consistent with the publicly accessible California's real time traffic data.

Index Terms— **Fluid Model, Fundamental Diagram, Newell, Unification**

I. Introduction

In our findings we demonstrate through existing real time field data from publicly accessible California Department of Transportation Performance Measurement System (PeMS) [1] that highway flow, to a notable extent, is governed by the predictability of human behavior in the confines of freeway and is independent of all other variables that are currently being applied in some of the highly complex parametric models used for analyzing traffic.

This study presents a mathematical model for traffic flow theory that is based on a singular variable: the driver behavior. We claim that it is the only required element essential for formulating and predicting all traffic patterns; that all drivers, either mostly adhere to the posted legal speed limits, or when necessary, adjust their speed to maintain a constant time of impact with the vehicle in front of them.

The retained time of impact varies from region to region, nevertheless it remains constant regionally. This constancy dictates both the capacity and the flow of the freeway [2].

The central assumption of this paper is based on our observations that the driver mode of behavior within the confines of the highway is to a great degree predictable; that the autonomous human agency, behind the wheel of an automobile, is forced to behave with a logical uniformity that is capable of preventing collision almost all the time (car accidents are rare occurrences.)

In lower traffic density situations (vehicular density $n < n_c$ critical density, measured in vehicles/mile) the driver behavior is governed by the posted speed limit, and the vehicle's speed is maintained close to the speed limit. Under more congested conditions (vehicular density $n > n_c$ critical density), the speed is lowered and continuously adjusted by the driver to preserve an approximately 2 seconds of impact time with the vehicle ahead, therefore the behavior is controlled by the desire to avoid collision. Figure 1 depicts that the measured traffic data fall on loci of a particular time of impact contour.

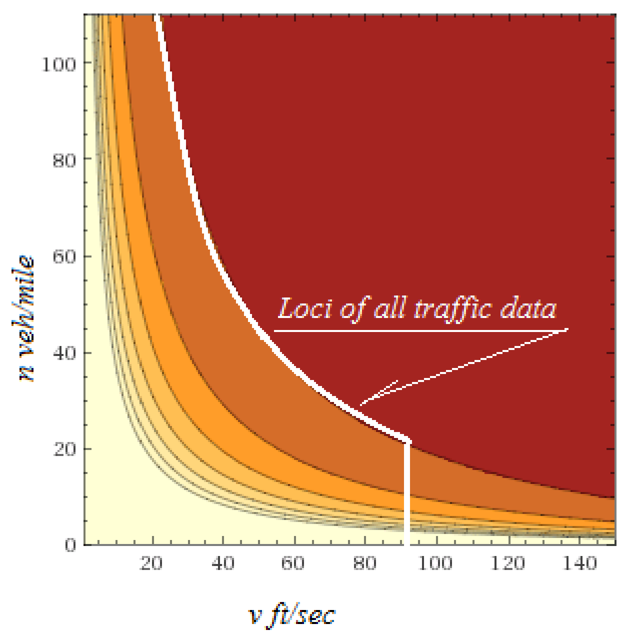

Figure 1: The plot of the loci of measured traffic data

We assert that he driver's adherence to a constant impact time regardless of the distance between two consecutive vehicles leads to the observed traffic patterns verifiable by the real time field data from aforementioned PeMS. Our simple assumption of predictability of driver behavior allows our calculus to replicate complete highway traffic flow (consistent with the Newell's diagram) and demonstrates the ability to predict traffic patterns in situations of lane convergence, on-ramp influx, lane drop and lane additions, congestion wave back propagation and frontal wave dispersion, in short it provides a model for the entire traffic flow field. We will also demonstrate that highway capacity is governed by the vehicular length and the prevalent speed and the impact time rather than the dimensions of the roadway. We will also demonstrate that a freeway may operate unstably for a short time at flow rates above the capacity and inevitably collapse into the congested but stable state at the slightest disturbance. (Lyoponov instability amplifies the effect of one tap on the breaks.)

II. Review of current literature

Many attempts to specify a relation among traffic flow, density, and speed have been made. A macroscopic traffic model to correlate them forms the so-called fundamental diagrams of traffic flow. Greenshields [3] derived a parabolic fundamental diagram between flow and density. Lighthill, Whitham [4] and Richards [5] used Greenshields' hypothesis and a conservation law of vehicles to provide a concave fundamental diagram, which is called the first order LWR model.

Newel [6] proposed a triangular flow-density fundamental diagram as a simpler alternative to solve the LWR model. Only two velocities characterize this model: a maximum free-flow velocity in a free-flow regime and a propagation velocity for a congestion area. However, he did not explain the correlation between propagation velocity and driving patterns. Banks [7] considered the time-gap which is the time required to travel the distance between the front-end of a vehicle to the back-end of its leading vehicle, and showed the relation between the time-gap and speed using real traffic data in the USA. He found that during congested times, the average time-gap is relatively constant, while it diverges with large deviation during free-flow periods. However, he did not derive a fundamental diagram from the time-gap nor an analysis method to estimate the time-gap from raw traffic data. Cho, Cruz, Rao and Badii derived an analytic equation for the speed of freeway as a function of density in the congested state consistent with Newell's diagram.

III. The traffic model

For the sake of continuity in argument, Let us define density n in the English system as

$$n = \frac{5280}{L_{veh} + D}$$

where 5280 is the number of feet in a mile, L_{veh} is the typical vehicular length, and D is inter vehicular distance, in feet.

Solving for D and dividing by the speed v, we can infer that for every density n there is a corresponding speed v measured in ft/sec and headway τ measured in seconds such that

$$\tau = \frac{D}{v} = \frac{5280 - nL_{veh}}{nv}$$

subject to:

$$\frac{\partial v}{\partial n} = 0\,, v = v_f \qquad \forall\, n < n_c$$

which imposes the legal speed limit for values of density below a critical density and

$$\frac{\partial \tau}{\partial n} = 0, \qquad v = \frac{5280 - nL_{veh}}{n\tau} \qquad \forall\, n > n_c$$

which imposes safe driving headway at constant τ. In the above equation v_f is the forward speed and may be interpreted as legally allowable speed limit. Systematic measurements of the headway τ, affirms that the value of τ very rapidly converges to a constant.

The only unknown in the above equations is n_c which will be derived from intersecting the line $q = nv_f$, where q is the flow volume measured in vehicles per hour with $q = n\left[\frac{5280 - nL_{veh}}{n\tau\bar{c}}\right]$ and solving for n, where $\bar{c}$=5280/3600 is the conversion factor from mph to ft/sec.

$$n_c = \frac{5280}{L_{veh} + v_f\tau\bar{c}}$$

The maximum capacity of the freeway per lane will then be $q_{max} = n_c v_f$. More important observation may be made that the downward slope of $q = \left[\frac{5280 - nL_{veh}}{n\tau\bar{c}}\right] n$ computes to be $\left[-\frac{L_{veh}}{\tau}\right]$.

It is noteworthy that this derivation is universal and applies to any roadway regardless of condition, and since the roadway capacity is a function v_f and τ, it can dramatically vary, for example in a situation like NASCAR where v_f is very large and τ is very small, the capacity is more than 20,000 veh/hr.

Also the freeway has been observed to operate above the capacity; this is due to a temporary smaller τ among a group of vehicles and the lack of v_f enforcement. Regardless of the reasons, the time gap based model is inherently unstable (in the Lyoponov sense of stability) and temporary, and it will inevitably collapse to the congested region of Newell diagram. It is also noteworthy that the reason that NASCAR is able to retain stability is due to the closed loop course. Perry Y. Li and Ankur Shrivastava [8], while studying a policy for automated cruise control investigated the stability of time gap based model and concluded that it was unstable on an open course and stable on a closed loop.

Applying our assertions (Cho, Cruz, Rao and Badii) to the LWR wave equation we may write the number of vehicles N in the system between two measuring stations A at upstream and B at downstream may be written as

$$\frac{dN(t)}{dt} = q_B(B,t) - q_A(A,t)$$

On the other hand we may express N in terms of density

$$N(t) = \int_A^B n(x,t)dt$$

Substituting

$$\frac{dN(t)}{dt} = \frac{d}{dt}\int_A^B n(x,t)dt = q_B(B,t) - q_A(A,t)$$

by Leibniz axiom we may write $q_B(B,t) - q_A(A,t)$ as

$$\int_A^B \frac{\partial}{\partial x} q(x,t)dx$$

Substituting

$$\frac{d}{dt}\int_A^B n(x,t)dt = \int_A^B \frac{\partial}{\partial x} q(x,t)dx$$

Dropping the integrals, we may write the above as

$$\frac{\partial n(x,t)}{\partial t} - \frac{\partial q(x,t)}{\partial x} = 0$$

But $q(x,t) = n(x,t)v(n)$ with v itself having dependence on n, i.e: $v = v(n(x,t))$

$$\frac{\partial n(x,t)}{\partial t} - v(n)\frac{\partial n(x,t)}{\partial x} - n(x,t)\frac{\partial v(n)}{\partial x} = 0$$

Expanding the last term

$$\frac{\partial n(x,t)}{\partial t} - v(n)\frac{\partial n(x,t)}{\partial x} - n(x,t)\frac{\partial v(n)}{\partial n}\frac{\partial n(x,t)}{\partial x} = 0$$

Substituting

$$v(n) = max\left[v_f, \frac{5280 - nL_{veh}}{n\tau}\right]$$

And

$$\frac{\partial v(n)}{\partial n} = min\left[0, \frac{5280}{n^2\tau}\right]$$

Applying the two flow regime concepts

$$\frac{\partial v}{\partial n} = 0 \; and \; v = v_f \; \forall \, n < n_c$$

we end up with a wave equation of the form

$$\frac{\partial n(x,t)}{\partial t} - [v_f]\,\frac{\partial n(x,t)}{\partial x} = 0 \qquad \forall \, n < \, n_c$$

which describes a density wave travelling in forward direction with the speed $[v_f]$

And as for the case where

$$v = \frac{5280 - nL_{veh}}{n\tau} \qquad \forall \, n > n_c$$

We may write

$$\frac{\partial n}{\partial t} - \left[\frac{5280 - nL_{veh}}{n\tau} - \frac{5280}{n^2\tau}n\right]\frac{\partial n}{\partial x} = 0$$

Leading to

$$\frac{\partial n(x,t)}{\partial t} + \left[\frac{L_{veh}}{\tau}\right]\frac{\partial n(x,t)}{\partial x} = 0 \qquad \forall \, n > \, n_c$$

which describes a wave travelling at speed of $\left[\frac{L_{veh}}{\tau}\right]$ backward, consistent with observations of scientists at the Nagoya University in Japan [9]. The appearance of $\left[\frac{L_{veh}}{\tau}\right]$ unifies the Newell model with LWR fluid model.

Although traffic waves travelling forward and backward are interesting as a phenomenon, they are of little value to practitioners of traffic engineering who deal on day to day basis with congestion. However we can take advantage of the constancy of impact time $\tau = \frac{D}{v}$ and provide a new methodology for day to day traffic engineering problems that have confounded traffic engineers since the dawn of congestion, for the past eighty years. Toward this goal we introduce the state equation for states α and β for traffic conditions where there is a change of density from n_α to n_β $\forall \, n$

If $n_\beta < n_c$ then:

$$\frac{dv}{dn} = 0\,, v = v_f\,; q = n_\beta v_f$$

However if $n_\beta > n_c$ then the inter-vehicular distance D_β for the state β

$$D_\beta = \frac{5280 - n_\beta L_{veh}}{n_\beta}$$

And by assertion of constancy of τ we may write

$$\tau = \frac{D_\alpha}{v_\alpha} = \frac{D_\beta}{v_\beta}$$

Eliminating τ we may write the transition of speed from state α to β and vice versa

$$v_\beta = v_\alpha \frac{D_\beta}{D_\alpha}$$

The immediate application of our findings to day to day traffic engineering is presented below.

A. Merging Traffic

An on-ramp traffic is merged with outermost lane. The merging occurs at the speed of the outermost lane with q_r representing on-ramp volume and q_1 and v_1 representing the volume and the speed on the outermost freeway lane. The density of the outermost lane can be calculated

$$n_1 = \frac{q_1}{v_1}$$

The inter vehicular distance in the outermost lane

$$D_1 = \frac{5280 - n_1 L_{veh}}{n_1}$$

The density of the ramp

$$n_r = \frac{q_r}{v_1}$$

The total density at the merge is then

$$n_{merg} = n_1 + n_r$$

When $n_{merg} > n_c$ one may compute the inter vehicular distance at merging point

$$D_{merg} = \frac{5280 - n_{merg} L_{veh}}{n_{merg}}$$

The speed after merging

$$v_{merg} = v_1 \frac{D_{merg}}{D_1}$$

The flow after merging

$$q_{merg} = (n_1 + n_r) v_{merg} = n_{merg} v_{merg}$$

Similarly the problem of contraction or expansion in the number of lanes may be addressed.

B. Lane Convergence

Let N_u and N_d represent the number of lanes upstream and downstream of a section. The resulting density after the transition

$$n_d = \frac{N_u}{N_d} n_u$$

where n_d and n_u are downstream density and upstream density values.

When $n_d > n_c$, one may compute the inter vehicular distance at merging point

$$D_d = \frac{5280 - n_d L_{veh}}{n_d}$$

and downstream speed as

$$v_d = v_u \frac{D_d}{D_u}$$

Then the downstream flow

$$q_d = n_d v_d$$

The traffic engineering practitioners may be more interested in calculating the time of congestion building. The flow rate at any segment in the system is

$$Q = (q_{upstream} - q_{downstream})$$

The number of vehicles accumulating per unit time (here in seconds) will be

$$N = \frac{Q}{3600}$$

The distance accumulated in unit time is then

$$\frac{dx}{dt} = N(L + D_d)$$

Substituting the corresponding values

$$\frac{dx}{dt} = \frac{(q_{upstream} - q_{downstream})(L_{veh} + D_d)}{3600}$$

By definition

$$(L_{veh} + D_d) = \frac{5260}{n_d}$$

Substituting

$$\frac{dx}{dt} = \frac{(q_{upstream} - q_{downstream})\bar{c}}{n_d}$$

$$\frac{dx}{dt} = \frac{v_{downstream}(q_{upstream} - q_{downstream})\bar{c}}{q_{downstream}}$$

Then the distance x at which the upstream space is depleted in time t

$$x = \frac{1}{N_u} N(L_{veh} + D_d) \times t$$

Replacing $(L_{veh} + D_d)$ and N with their equivalents, the rate of depletion of space will be proportional to the speed of upstream vehicles and is

$$\frac{dx}{dt} = \frac{\bar{c}}{N_u} \frac{(q_{upstream} - q_{downstream})}{n_d}$$

$$= \frac{v_{downstream}(q_{upstream} - q_{downstream})\bar{c}}{N_u q_{downstream}}$$

The time in seconds for the upstream densities, and speed to equalize with downstream density and speed is

$$t = \frac{3600 X N_u q_{downstream}}{v_{downstream}(q_{upstream} - q_{downstream})}$$

where X, the segment length, is the distance between the upstream and the downstream measurement stations in miles. Conversely, when $Q = q_{upstream} - q_{downstream}$ is a negative number, it implies a reduction in density as a result of net gain in space.

IV. Conclusion

We derived the Newell diagram from a simple tenet of human behavior. Applying our results we were able to compute the capacity of the roadway. Also we derived the LWR equation for both regions of Newell diagram consistent with observation by applying the driver behavior. We also referred that the human behavior is inherently unstable and unsuitable to mimic for the purposes of designing automated cruise control as is suggested by Berthold Horn of MIT [10] and other researchers.

We provided a simple computational method for day to day traffic engineering calculations that is new and consistent with traffic observations.

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